Lecture 11

11.2. The Parallel Oscillatory Circuit

11.2.1. Resonance Condition and Resonance Frequency

The parallel oscillatory circuit is a circuit in which branches with an inductance and a capacitance are connected in parallel with a source of signal $\dot{E}_m$ (Fig. 11.10).

Determine the conductance of the branches containing an inductance and a capacitance:

$$Y_L = \frac{1}{r_1 + jx_L} = \frac{1}{r_1^2 + x_L^2} - j\frac{x_L}{r_1^2 + x_L^2} = g_L - jb_L; \qquad (11.23)$$

$$Y_C = \frac{1}{r_2 - jx_C} = \frac{1}{r_2^2 + x_C^2} + j\frac{x_C}{r_2^2 + x_C^2} = g_C + jb_C. \qquad (11.24)$$

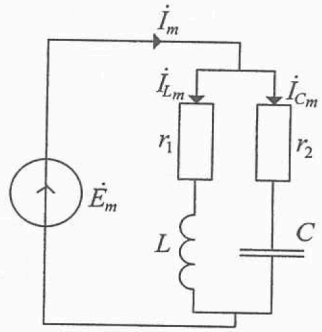

Fig. 11.10

The phenomenon where a circuit conductance becomes purely active is called resonance. The circuit susceptance at resonance is equal to zero:

$$b = -b_L + b_C = 0. \qquad (11.25)$$

Keeping in mind that at the resonance frequency ω_p

$$x_{Lp} = \omega_p L, \; x_{Cp} = \frac{1}{\omega_p C}, \; \rho = \sqrt{\frac{L}{C}}, \quad \omega_0 = \frac{1}{\sqrt{LC}},$$

from (11.23)–(11.25) we get:

$$\frac{\omega_p L}{r_1^2 + \omega_p^2 L^2} = \frac{1}{\omega_p C\left(r_2^2 + \dfrac{1}{\omega_p^2 C^2}\right)}.$$

Multiplying and dividing the reactance by ω_0:

$$\frac{\omega_0 L \dfrac{\omega_p}{\omega_0}}{r_1^2 + \omega_0^2 L^2 \dfrac{\omega_p^2}{\omega_0^2}} = \frac{1}{\omega_0 C \dfrac{\omega_p}{\omega_0}\left(r_2^2 + \dfrac{1}{\omega_0^2 C^2 \dfrac{\omega_p^2}{\omega_0^2}}\right)}$$

or

$$\frac{\rho\frac{\omega_p}{\omega_0}}{r_1^2+\rho^2\frac{\omega_p^2}{\omega_0^2}} = \frac{\rho}{\frac{\omega_p}{\omega_0}\left(r_2^2+\frac{\rho^2}{\frac{\omega_p^2}{\omega_0^2}}\right)} = \frac{\rho\frac{\omega_p^2}{\omega_0^2}}{\frac{\omega_p}{\omega_0}\left(\rho^2+r_2^2\frac{\omega_p^2}{\omega_0^2}\right)} = \frac{\rho\frac{\omega_p}{\omega_0}}{\rho^2+r_2^2\frac{\omega_p^2}{\omega_0^2}}.$$

Now we get

$$\frac{1}{r_1^2+\rho^2\frac{\omega_p^2}{\omega_0^2}} = \frac{1}{\rho^2+r_2^2\frac{\omega_p^2}{\omega_0^2}}$$

or

$$r_1^2+\rho^2\frac{\omega_p^2}{\omega_0^2} = \rho^2+r_2^2\frac{\omega_p^2}{\omega_0^2}.$$

Hence

$$\frac{\omega_p^2}{\omega_0^2}\left(r_2^2-\rho^2\right)=r_1^2-\rho^2.$$

And finally, the resonance frequency is:

$$\omega_p=\omega_0\sqrt{\frac{r_1^2-\rho^2}{r_2^2-\rho^2}}.$$

As it takes place, the following conditions are distinguished:

1) for $r_1 \ll \rho$, $r_2 \ll \rho$, we get $\omega_p=\omega_0=\frac{1}{\sqrt{LC}}$, that is the resonance frequency is equal to the natural frequency of the oscillatory circuit;

2) for $r_1=r_2$ we also get $\omega_p=\omega_0=\frac{1}{\sqrt{LC}}$;

3) for $r_1>\rho$ and $r_2>\rho$ or $r_1<\rho$ and $r_2<\rho$ a resonance occurs at the frequency $\omega_p \neq \omega_0$;

4) for $r_1=r_2=\rho$ expression (11.27) gives an uncertainty. A resonance takes place at any frequency. Such a resonance is called "neutral".

In general, in a parallel oscillatory circuit the resonance frequency ω_p depends on the active resistances in the circuit branches.

11.2.2. Complex Functions and Basic Frequency Characteristics of a Parallel Oscillatory Circuit

Consider the complex input impedance of a circuit:

$$Z_e(j\omega)=\frac{\dot{E}_m}{\dot{I}_m}=\frac{(r_1+j\omega L)\left(r_2+\frac{1}{j\omega C}\right)}{r_1+j\omega L+r_2+\frac{1}{j\omega C}}.$$

Taking into account that

$$r_1r_2 \ll \frac{L}{C}=\rho^2,\ \frac{r_1}{\omega C} \ll \frac{L}{C}=\rho^2,\ r_2\omega L \ll \frac{L}{C}=\rho^2 \quad \text{and} \quad r=r_1+r_2,$$

we get, considering (11.11):

$$Z_e(j\omega)=\frac{L}{C}\frac{1}{r+j\left(\omega L-\frac{1}{\omega C}\right)}=\frac{\rho^2}{r}\frac{1}{1+j\frac{x}{r}}=\frac{Q^2}{1+j\xi}.$$

At resonance ($\xi=0$): $Z_e(j\omega)=Q^2r$.

That is, at resonance the equivalent input impedance of a circuit has the active component only, and it is Q^2 times more than the active resistance in the circuit branches.

The normalized input impedance is:

$$Z_e(j\omega)=Z_{en}e^{j\varphi_e(\omega)}=\frac{Z_e(j\omega)}{Z_e(j\omega_0)}=\frac{1}{1+j\xi},$$

where

$$Z_{en}(\omega)=\frac{1}{\sqrt{1+\xi^2}}=\frac{1}{\sqrt{1+Q^2\nu^2}}=$$

$$=\frac{1}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}\approx\frac{1}{\sqrt{1+4Q^2\delta^2}}, \qquad (11.26)$$

$$\varphi_e(\omega)=-\mathrm{atan}\,\xi=-\mathrm{atan}\,Q\nu=-\mathrm{atan}\,Q\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\approx-\mathrm{atan}\,2Q\delta. \qquad (11.27)$$

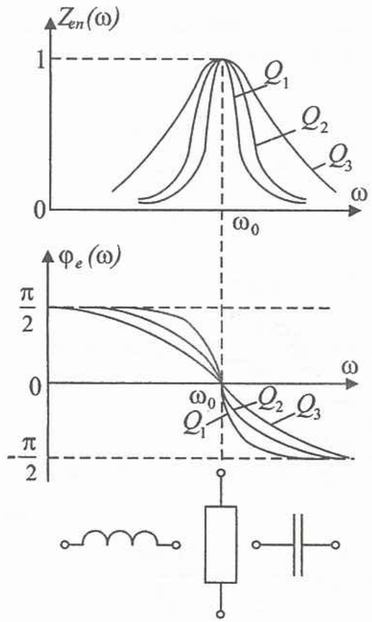

Fig. 11.11

The diagrams of the amplitude-frequency characteristic according to (11.26) and of the phase-frequency characteristic according to (11.27) are given in Fig. 11.11. Here, $Q_1 > Q_2 > Q_3$. Compared with Fig. 11.2, the phase frequency characteristic has changed its sign. At $\omega < \omega_0$ the capacitive reactance is greater than the inductive reactance, and it can be neglected at $\omega \to 0$, that is the branch with a capacitance can just be broken. As a result, we have a branch with an inductance, and the circuit is of inductive nature. Similarly for $\omega > \omega_0$, where we can break the branch with the inductance, the circuit is of capacitive nature. Define the currents in the circuit branches I_{mL0}, I_{mC0} and the source current at resonance.

As the impedances of the branches with the inductance and capacitance are

$$Z_{L0} \approx \omega_0 L = \rho, \quad Z_{C0} \approx \frac{1}{\omega_0 C} = \rho,$$

then

$$I_{Lm0} = I_{Cm0} = \frac{E_m}{\rho} = \frac{E_m}{Qr}.$$

Taking into account the equivalent resistance of the circuit, we get:

$$I_{m0} = \frac{E_m}{Q^2 r}$$

or

$$I_{Lm0} = I_{Cm0} = QI_0.$$

In other words, the currents in the branches are Q times greater than the source current. Therefore, resonance in a parallel oscillatory circuit is called current resonance.

Consider the frequency characteristics of a parallel oscillatory circuit connected to a real voltage source (Fig. 11.12).

Determine the frequency characteristics in terms of current. The normalized source current is:

$$I_{mn}(j\omega) = \frac{I_m(j\omega)}{I_m(j\omega_0)} = \frac{\dot{E}_m Y(j\omega)}{\dot{E}_m Y(j\omega_0)} = \frac{Y(j\omega)}{Y(j\omega_0)}, \qquad (11.28)$$

where $Y(j\omega), Y(j\omega_0)$ are the complex input conductance of the circuit at any frequency and at resonance.

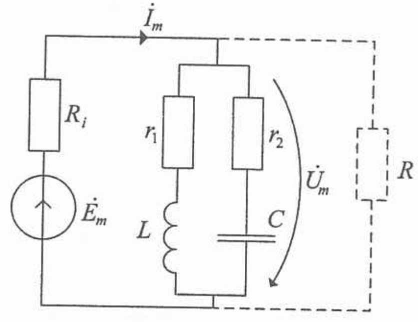

Fig. 11.12

According to Fig. 11.12

$$Y(j\omega) = \frac{1}{R_i + Z_e(j\omega)} = \frac{1}{R_i + \frac{R_{e0}}{1+j\xi}} = \frac{1+j\xi}{R_i + R_{e0} + j\xi R_i} =$$

$$= \frac{1+j\xi}{(R_i + R_{e0})\left(1 + j\xi\frac{R_i}{R_i + R_{e0}}\right)} = \frac{1+j\xi}{(R_i + R_{e0})(1+i\xi_e)}, \qquad (11.29)$$

where $R_{e0} = Q^2 r$ is the equivalent resistance of the circuit at resonance; R_i is the internal resistance of the voltage source; $\xi_e = \xi\frac{R_i}{R_i + R_{e0}} = \nu Q\frac{R_i}{R_i + R_{e0}} = Q_e \nu$ is the equivalent generalized detuning;

$Q_e = Q\frac{R_i}{R_i + R_{e0}} = \frac{\rho}{r}\frac{1}{1 + \frac{\rho^2}{rR_i}}$ is the equivalent Q-factor of the circuit.

At the resonance frequency:

$$Y(j\omega_0) = \frac{1}{R_i + R_{e0}}.$$

Then according to (11.28) and (11.29)

$$I_{mn}(j\omega) = I_{mn} e^{j\varphi_i(\omega)} = \frac{1+j\xi}{1+j\xi_e},$$

where

$$I_{mn}(\omega)=\sqrt{\frac{1+j\xi^2}{1+j\xi_e^2}}=\sqrt{\frac{1+Q^2\nu^2}{1+Q_e^2\nu^2}}=\sqrt{\frac{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}{1+Q_e^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}\approx\sqrt{\frac{1+4Q^2\delta^2}{1+4Q_e^2\delta^2}};$$

$$\varphi_i(\omega)=\text{atan}\xi-\text{atan}\xi_e=\text{atan}Q\nu-\text{atan}Q_e\nu=$$

$$=\text{atan}Q\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)-\text{atan}Q_e\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\approx\text{atan}2Q\delta-\text{atan}2Q_e\delta.$$

The frequency characteristics in terms of current are given in Fig. 11.13.

Since the argument here is the detuning ξ, these curves are called the resonance characteristics. In Fig. 11.13 we can see that the resonance properties of a parallel circuit in terms of current are expressed stronger if the internal resistance R_i of the signal source decreases.

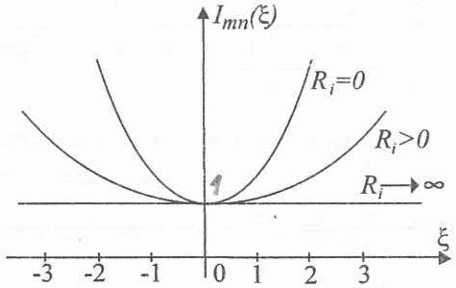

Fig. 11.13

Determine the frequency characteristics in terms of voltage.

The normalized voltage across the circuit is:

$$U_{mn}(j\omega)=\frac{U_m(j\omega)}{U_m(j\omega_0)}. \tag{11.30}$$

The voltage across the circuit is:

$$U_m(j\omega)=I_m(j\omega)Z_e(j\omega)=\dot{E}_mY(j\omega)Z_e(j\omega).$$

Using (11.29), we get

$$U_m(j\omega)=\dot{E}_m\frac{1+j\xi}{(R_i+R_{e0})(1+j\xi_e)}\frac{Q^2r}{1+j\xi}=\frac{\dot{E}_mQ^2r}{R_i+R_{e0}}\frac{1}{1+j\xi_e}.$$

At the resonance frequency ($\xi_e=0$):

$$U_m(j\omega_0)=\frac{\dot{E}_mQ^2r}{R_i+R_{e0}}. \tag{11.31}$$

Then, according to (11.30) and (11.31)

$$U_m(j\omega)=U_{mn}(\omega)e^{j\varphi_u(\omega)}=\frac{1}{1+j\xi_e},$$

where

$$U_{mn}(\omega)=\frac{1}{\sqrt{1+\xi_e^2}}=\frac{1}{\sqrt{1+Q_e^2\nu^2}}=$$

$$=\frac{1}{\sqrt{1+Q_e^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}\approx\frac{1}{\sqrt{1+4Q^2\delta^2}};$$

$$\varphi_u(\omega)=-\text{atan}\xi_e=-\text{atan}Q_e\nu=-\text{atan}Q_e\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\approx-\text{atan}2Q_e\delta.$$

The resonance characteristics in terms of voltage are given in Fig. 11.14.

We can see from Fig. 11.14 that the resonance properties of a parallel oscillatory circuit in terms of voltage are expressed stronger if the internal resistance R_i of the signal source increases.

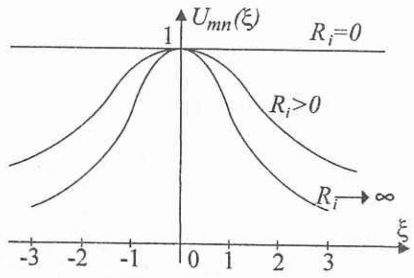

Fig. 11.14

11.2.3. Bandwidth and Bandwidth Shape Factor (Bandwidth Ratio)

We will determine the bandwidth of a parallel oscillatory circuit by analogy with a series oscillatory circuit, using the expression for the voltage amplitude-frequency characteristic:

$$U_{mn}(\omega)=\sqrt{\frac{1}{1+j\xi_e^2}}=\frac{1}{\sqrt{2}}.$$

After transformations similar to those in 11.1.4, we obtain for the bandwidth:

$$B=B_{0,707}=\frac{\omega_0}{Q_e};\quad B_{0,1}=\frac{10\omega_0}{Q_e};\quad B_0=\frac{1}{Q_e}=d_e.$$

The bandwidth ratio (shape factor) is: $K_B=0{,}1$.

That is, the parallel and series oscillatory circuits have identical bandwidth ratios.

We can see that the bandwidth is determined by the Q-factor of a circuit Q_e, i.e. with constant primary parameters of the r, L, C-circuit it is determined by the internal resistance R_i of the signal source which by-passes the circuit. To extend the bandwidth, it is possible to by-pass the circuit additionally by the resistance R_{shunt} (Fig. 11.12).

Then, the total resistance bypassing the circuit is:

$$R_{gen}=\frac{R_i R_{shunt}}{R_i+R_{shunt}}.$$

As a result the equivalent Q-factor is:

$$Q_e=Q\frac{R_{gen}}{R_{gen}+R_{e0}}.$$

The bandwidth is:

$$B_0=\frac{1}{Q_e}=\frac{1}{Q}\left(1+\frac{R_{e0}}{R_{gen}}\right)=\frac{r}{\rho}+\frac{\rho}{R_{gen}}=$$

$$=\frac{r}{\rho}+\frac{\rho(R_i+R_{shunt})}{R_i\cdot R_{shunt}}=\frac{r}{\rho}+\frac{\rho}{R_i}+\frac{\rho}{R_{shunt}}.$$

11.2.4. Complex Parallel Oscillatory Circuits

If at least one branch of a parallel oscillatory circuit (Fig. 11.9) has two heterogeneous reactances, the circuit is called a complex parallel oscillatory circuit.

There are four types of complex circuits (Fig. 11.15, *a–d*). Such circuits make it possible to reduce the by-passing effect of the internal impedance of the signal source and that of the load impedance on the circuit and thus to match the equivalent impedance of the circuit with the load resistance. Besides, such circuits allow suppressing other frequencies.

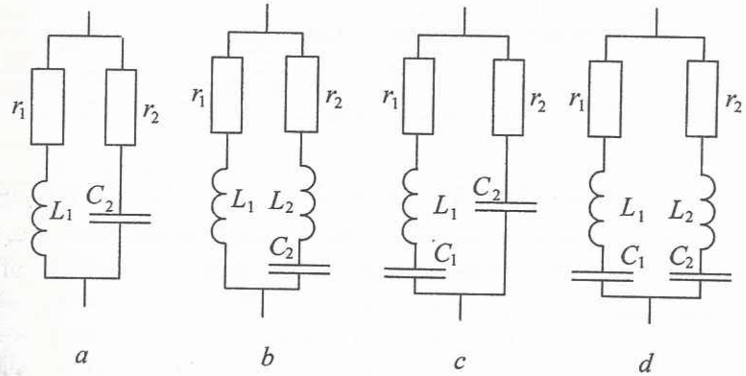

Fig. 11.15

Consider an oscillatory circuit of the 4-th type, which is the most general of the four types (Fig. 11.15, *d*).

The resonance condition in such a circuit, as well as in a circuit of the 1-st type (Fig. 11.15, *a*), is the total admittance of the circuit being equal to zero.

Neglecting the active resistances r_1 and r_2 for high-Q circuits, we get:

$$\frac{1}{x_1}+\frac{1}{x_2}=0$$

or

$$x_1+x_2=0.$$

That is

$$\omega_p L_1 - \frac{1}{\omega_p C_1} + \omega_p L_2 - \frac{1}{\omega_p C_2} = \omega_p (L_1 + L_2) - \frac{1}{\omega_p}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) =$$

$$= \omega_p L - \frac{1}{\omega_p C} = 0,$$

where $L = L_1 + L_2$; $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$; $C = \frac{C_1 C_2}{C_1 + C_2}$ — equivalent inductance and capacitance of the circuit.

Hence, the secondary parameters of the circuit:

$$\omega_p \approx \omega_0 = \frac{1}{\sqrt{LC}}; \quad \rho = \sqrt{\frac{L}{C}}; \quad Q = \frac{\rho}{r} = \frac{1}{d}; \quad r = r_1 + r_2$$

Determine the equivalent resistance of the circuit R_{e_0} at resonance.

Since the equivalent resistance of the circuit is of active nature, it is determined as the sum of the active conductances g_1 and g_2 of the circuit branches

$$R_{e_0} = \frac{1}{g_1 + g_2}, \tag{11.32}$$

where

$$g_1 = \frac{r_1}{r_1^2 + x_1^2}, \quad g_2 = \frac{r_2}{r_2^2 + x_2^2}. \tag{11.33}$$

Neglecting r_1 and r_2 in the denominator as compared with x_1, x_2, from (11.32) and (11.33) we obtain:

$$R_{e0} = \frac{1}{\frac{r_1}{x_1^2} + \frac{r_2}{x_2^2}} = \frac{x_1^2 x_2^2}{r_1 x_2^2 + r_2 x_1^2}.$$

Since

$$x_1 = -x_2,$$

then

$$R_{e0} = \frac{x_1^2}{r} = \frac{x_2^2}{r} = \frac{1}{r}\left(\omega_0 L_1 - \frac{1}{\omega_0 C_1}\right)^2 = \frac{1}{r}\left(\omega_0 L \frac{L_1}{L} - \frac{1}{\omega_0 C_1}\right)^2 =$$

$$= \frac{(\omega_0 L)^2}{r}\left(\frac{L_1}{L} - \frac{1}{\omega_0^2 L C_1}\right)^2 = \frac{\rho^2}{r}\left(\frac{L_1}{L} - \frac{C}{C_1}\right)^2 = Q^2 r (m_L - m_C)^2, \tag{11.34}$$

where $m_L = \frac{L_1}{L} = \frac{L_1}{L_1 + L_2}$; $m_C = \frac{C}{C_1} = \frac{C_2}{C_1 + C_2}$ – inclusion coefficients of inductance and capacitance.

The coefficients m_L, m_C show what part of the common inductance L or of the common capacitance C of the circuit is included into one or the other branch of the circuit.

$$0 \le m_L \le 1, \quad 0 \le m_C \le 1.$$

As it follows from (11.34), the equivalent resistance R_{e0} of a complex circuit is considerably smaller than the equivalent resistance $Q^2 r$ of a first-type circuit.

Therefore, a complex circuit allows shunting by using a considerably smaller load resistance with no significant expansion of the bandwidth and no deterioration of the selective properties of the circuit.

Hence, the equivalent resistance of the circuit can be varied within a wide range (by varying the inclusion coefficients m_L, m_C) without changing the circuit parameters Q, ρ, ω_0.

Evidently, for a first-type circuit $m_L = 1$, $m_C = 0$, for a second-type circuit $m_C = 0$, a third-type circuit $m_L = 1$.

Fig 11.15 shows that the branches of a complex parallel oscillatory circuit are individual series oscillatory circuits in which additional voltage resonances are possible at the frequencies:

$$\omega_{01} = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{m_L L \frac{C}{m_C}}} = \omega_0 \sqrt{\frac{m_C}{m_L}};$$

$$\omega_{02} = \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\sqrt{(1 - m_L) L \frac{C}{1 - m_C}}} = \omega_0 \sqrt{\frac{1 - m_C}{1 - m_L}}.$$

At these frequencies, the branch impedances and consequently the impedance of the whole circuit drop sharply.

The frequency characteristics of the circuits, taking into account the above mentioned resonances, are shows in Fig. 11.16.

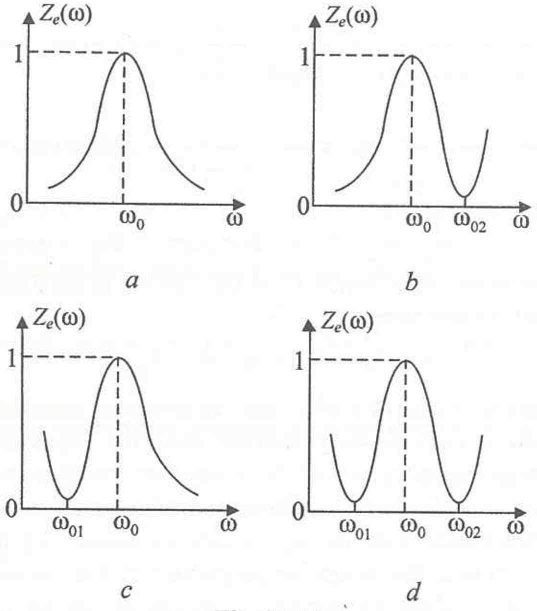

Fig. 11.16

11.2.5 Some Applications of Parallel Oscillatory Circuits

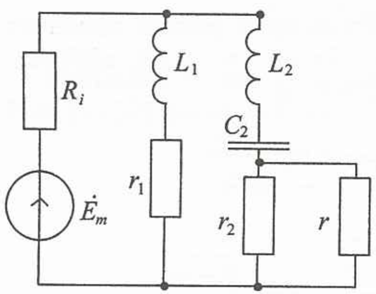

Fig. 11.17

A parallel oscillatory circuit is usually connected in parallel to the load (see Fig. 11.12). To reduce the influence of the load resistance, complex parallel circuits are used (Fig. 11.17).

A parallel oscillatory circuit can also be connected in series to the load, for example, in a rejection filter (rejector).

In such a filter (Fig. 11.18) at the resonance frequency, the circuit L_f, C_f suppresses interference signals coming together with the useful signal from the antenna A to the input oscillatory circuit LC of a high frequency amplifier (HFA). C_p is a blocking capacitor.

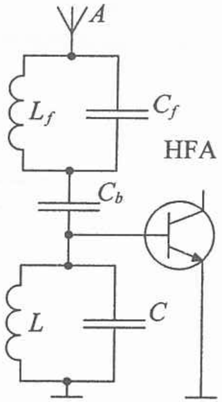

Fig. 11.18

Parallel oscillatory circuits can be connected in series to one another for suppressing several signals of different frequencies.

So, in a harmonic filter (Fig. 11.19) the parallel oscillatory circuits $L_{f_1}, C_{f_1}, \ldots, L_{f_k}, C_{f_k}$ do not pass signals of the frequencies

$\omega_1 = \dfrac{1}{\sqrt{L_{f_1} C_{f_1}}}, \ldots, \omega_k = \dfrac{1}{\sqrt{L_{f_k} C_{f_k}}}$ to the load r_l.

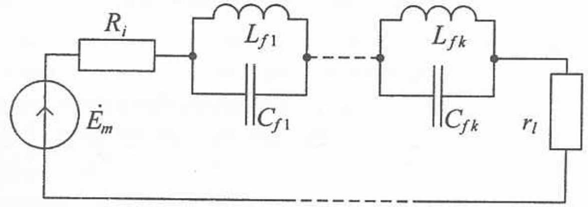

Fig. 11.19

Example 3

Determine the resonance frequency f_0, the characteristic impedance ρ, the quality factor Q and the resonance resistance R_0 of a parallel oscillatory circuit (Fig. 11.20) if $L = 0{,}2$ mH; $R = 12\ \Omega$; $C = 360$ pF.

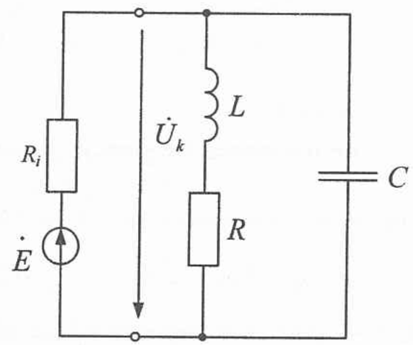

Fig. 11.20

Solution

In the case of small losses, the resonance frequency, the characteristic impedance and the quality factor Q of a parallel oscillatory circuit are determined by the expressions

$$f_0 = \frac{1}{2\pi\sqrt{LC}} =$$

$$= \frac{1}{2\pi\sqrt{0,2\cdot10^{-3}\cdot360\cdot10^{-12}}} = 593 \text{ kHz};$$

$$\rho = \sqrt{\frac{L}{C}} = \sqrt{\frac{0,2\cdot10^{-3}}{360\cdot10^{-12}}} = 745\ \Omega;$$

$$Q = \frac{\rho}{R} = \frac{745}{12} = 62,1.$$

The resonance resistance R_0 of the parallel oscillatory circuit is Q times greater than its characteristic impedance:

$$R_0 = Q_\rho = 62,1\cdot745 = 46,3\ k\Omega.$$

Example 4

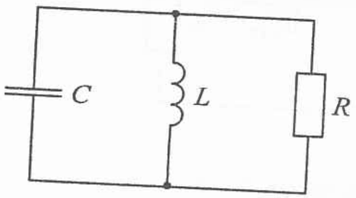

Fig. 11.21

Calculate the elements of a parallel oscillatory circuit (Fig. 11.21) with a bandwidth of $2\Delta f = 25$ kHz, the central bandwidth frequency (the resonance frequency) $f_0 = 1$ MHz and the characteristic impedance $\rho = \sqrt{\frac{L}{C}} = 75\ \Omega$.

Solution

The resonance frequency f_0 and the characteristic impedance ρ of the parallel oscillatory circuit are related as: $f_0 = \frac{1}{2\pi\sqrt{LC}}$; $\rho = \sqrt{\frac{L}{C}}$.

Solving these equations with respect to L and C, we get

$$C = \frac{1}{2\pi f_0\rho} = \frac{1}{2\pi\cdot1\cdot10^6\cdot75} = 2,12\cdot10^{-6} = 2,12 \text{ nF}$$

$$L = \frac{1}{2\pi f_0} = \frac{1}{2\pi\cdot1\cdot10^6} = 11,9\cdot10^{-6} = 11,9 \text{ mcH}.$$

The quality factor is related with the bandwidth and the resonance frequency as $2\Delta f_0 = \frac{f_0}{Q}$. Hence, $Q = \frac{f_0}{2\Delta f_0} = \frac{1\cdot10^6}{25\cdot10^3} = 40.$